Problems 5: Continuous Markov process and the diffusion equation

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Question 1

Give a definition of Markov stochastic process. What is a continuous Markov process?

Question 2

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random variables. Which of the following processes is Markov?

a) $y_n = x_1 + \cdots + x_n$

b) $y_n = x_1 \times \cdots \times x_n$

c) $y_n = \max\{0, x_1, \dots, x_n\}$

d) $y_n = \left(n, \frac{x_1+\cdots+x_n}{n}\right)$

Hint: convert to iteration $y_{n+1} = f(y_n)$.

Question 3

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random variables. Why is the following process not Markov?

$$y_n = [x_1 + \cdots + x_n]$$

where $[\cdot]$ means rational part.

Question 4

Let $\{m(t)\}_{t\geq 0}$ be a stochastic process defined by

$$m(t) = \frac{n(t) - \nu t}{\sqrt{\nu}}$$

where $n(t)$ is the value of a stationary Poisson process $\{n(t)\}_{t\geq 0}$ with expected value $\mathbb{E}\{n(t)\} = \nu t$ (see Appendix A). Use properties of the Poisson process to

a) Show that the expected value and variance of $m(t)$ are:

$$\mathbb{E}\{m(t)\} = 0\,, \qquad \sigma^2(m(t)) = t$$

b) Prove that the differential $dm(t) = m(t + dt) - m(t)$ has the property

$$\lim_{\nu\to\infty}(dm^2) = dt$$

c) The stochastic process $\{m(t)\}_{t\geq 0}$ becomes Wiener process as $\nu \to \infty$.

A Poisson process

The Poisson point process $\{n(t)\}_{t\geq 0}$ is a discrete-valued stochastic process counting the number $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ of occurrences of some event during time interval $[0, t]$, and satisfying the following properties:

Independent increments : The number $\Delta n(\Delta t) = n(t + \Delta t) - n(t)$ of events in the interval Δt is independent of the number of events in any other interval non-overlapping with Δt (e.g. $[0, t]$). This property implies $\{n(t)\}_{t\geq 0}$ is a Markov process.

Orderliness : The probability of two or more events during sufficiently small interval Δt is essentially zero:

$$P\{\Delta n(\Delta t) \geq 2\} = o(\Delta t)$$

(note the use of the small 'o' notation.)

These two properties imply that the number $n(t)$ of events in $[0, t]$ has Poisson distribution. For stationary (or homogeneous) Poisson process this distribution is

$$P(n(t)) = \frac{(\nu t)^n}{n!}\, e^{-\nu t}$$

where ν is the expected *rate* or *intensity* parameter (the expected number of events in a unit interval). The expected value and the variance for stationary Poisson process are respectively

$$\mathbb{E}\{n(t)\} = \nu t\,, \qquad \sigma^2(n(t)) = \nu t$$

The orderliness property implies also that differential $dn(t) = n(t+dt) - n(t)$ can have only two values: $dn(t) = 0$ (almost everywhere) or $dn(t) = 1$ (in a set of measure zero). This means that the differential dn of a Poisson process satisfies the property

$$dn^2 = dn$$